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Generalized Herfindahl-Hirschman Index to Estimate Diversity Score of a Portfolio across Multiple Correlated Sectors

Vaibhav Anand[1 2]
Ramasubramanian S V

Abstract

Portfolio diversification is an important risk mitigation strategy. However, the measurement of the degree of diversification in a portfolio may not be straightforward. Herfindahl-Hirschman Index is one of the most popular measure of diversification but the index does not account for the correlations across assets in a portfolio. We propose in this paper a more general and effective measure, the Generalized Herfindahl-Hirschman Index, to quantify the degree of diversification in a credit portfolio across multiple layers of correlated sectors and subsectors.

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[1]The authors work for IFMR Capital, Chennai.
[2]Email (correponding author): vaibhav.anand@ifmr.co.in
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Portfolio diversification is an important risk mitigation strategy for a credit portfolio manager. Diversification is a significant, and sometimes the only, source of risk mitigation and protection against large losses in case of event risks. It is important for the risk manager to *measure* and *monitor* the degree of diversification across various factors such as counterparties, sectors, and geographies etc. However, the quantification of diversification, a diversity score, in a portfolio may not be straightforward. The diversity score is defined as a *measure* of the degree of diversification for a given portfolio and *not* the optimum diversification level for a desired portfolio. We propose in this paper a more general and effective measure, the Generalized Herfindahl-Hirschman Index, to quantify the degree of diversification in a credit portfolio across multiple layers of correlated sectors and subsectors. In the first section, the classical Herfindahl-Hirschman Index formulation is discussed. In the next section, the key characteristics of an effective diversity score are discussed. In the subsequent sections, the Generalized Herfindahl-Hirschman Index formulation is derived and its interpretation as a measure of risk is discussed. The paper concludes with an implementation of the generalized approach on a hypothetical portfolio and with a comparison of its results with those of a classical approach.

Herfindahl-Hirschman Index (HHI)

One commonly used method of measuring the degree of diversification is the Herfindahl-Hirschman Index or HHI [1], named after the economists Orris C. Herfindahl and Albert O. Hirschman. Ávila et. al. used HHI in a detailed study to estimate the portfolio concentration using aggregate data [2]. HHI is defined as sum of the squares of the portfolio proportions. Consider a loan portfolio P with exposure across 3 counterparties, C_i, with corresponding proportions, c_i, where $i = 1$ to 3. Then the degree of diversification for P across counterparties can be measured using *HHI*, where

$$HHI = c_1^2 + c_2^2 + c_3^2 = \sum_{i=1}^{3} c_i^2 \tag{1}$$

HHI is effectively the weighted average of the proportions with proportions themselves as weights. If the proportions are equal in the portfolio P, i.e. $c_1 = c_2 = c_3 = 1/3$, then $HHI_{equal} = 3 \times (1/3)^2 = 1/3$. Similarly, if the proportions are unequal and skewed towards a single counterparty, say, $c_1 = 0.9$ and $c_2 = c_3 = 0.05$, then $HHI_{unequal} = 0.9^2 + 0.05^2 + 0.05^2 = 0.815$. If we take the reciprocal of the two *HHIs*, we get

$$\frac{1}{HHI_{equal}} = 3$$

and

$$\frac{1}{HHI_{unequal}} = \frac{1}{0.815} \approx 1.227.$$

The $\frac{1}{HHI}$ can be interpreted as the *effective* number of counterparty names in the portfolio [3], i.e. *three* in the proportionate and *nearly one* in the disproportionate portfolio. In this way, HHI is not only an indicator to measure the degree of diversification but its reciprocal has a more intuitive interpretation in terms of the *effective* number of units or entities across which the exposure is assumed.

Diversification across Sectors with Correlation

An effective measure of diversification should take into account the relation among the counterparties. A portfolio with multiple counterparties belonging to the same industry, say, banking, should have lower diversification score than that of an identical portfolio but where no two counterparties belong to the same industry. HHI provides a useful measure to quantify the diversification as a *score* if it is measured across a single level of sectors with zero correlation among them, i.e. across uncorrelated counterparties. It is not straightforward to estimate the diversification score using the same HHI measure taking into account the exposure across different correlated counterparties *(sub-sectors)* as well as their overlying industries *(sectors)*. For the sake of simplicity, it is assumed here onwards that there is a *one-to-many* mapping from sectors to subsectors, e.g. one industry may have more than one counterparty underlying it but not the other way round. Further it is assumed that sectors are uncorrelated but there could be a non-zero correlation among the subsectors within a sector, i.e. counterparties within the banking industry may be correlated but banking industry is assumed to be uncorrelated to real estate industry. Similarly, subsubsectors within a subsector may be correlated. An effective measure of diversification should take into account *(a)* subsectors within sectors and so on, as well as *(b)* correlations among the subsectors (and subsubsectors) in each sector.

Though HHI, in its original form, does not take into account multiple sectors and correlation, we show in the following sections that the above mentioned definition of HHI, i.e. weighted average of proportions, is a special case of a more generic formulation which is capable of addressing multiple layers of sectors and correlations among subsectors.

Portfolio Risk and Diversification

According to the classic Markowitz Portfolio Theory (MPT), the portfolio return variance (σ_p^2), a measure of portfolio risk, is defined as follows for a two asset portfolio [4]

$$\sigma_p^2 = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + 2c_1c_2\sigma_1\sigma_2\rho$$

where σ_1 and σ_2 are standard deviations of the two underlying assets, C_1 and C_2, ρ is correlation between the assets, and c_1 and c_2 are the respective proportions of the two assets in the portfolio. Since we are interested only in the estimation of degree of diversification irrespective of the riskiness of underlying assets in a portfolio, we can assume a special case where $\sigma_1 = \sigma_2 = \sigma$ for the above portfolio without the loss of generality. Here σ can be interpreted as proxy for the *average risk* in the underlying assets. Using this, we can write the above portfolio risk equation as

$$\frac{\sigma_p^2}{\sigma^2} = c_1^2 + c_2^2 + 2c_1c_2\rho \tag{2}$$

We discussed earlier that HHI assumes zero correlation among sectors (underlying assets here). Thus assuming zero correlation between the two assets, i.e. putting $\rho = 0$, we get

$$\frac{\sigma_p^2}{\sigma^2} = c_1^2 + c_2^2.$$

The above equation is similar to the equation for estimating the HHI (refer to equation 1) for a two asset portfolio, i.e.

$$HHI = \sum_{i=1}^{2} c_i^2 = c_1^2 + c_2^2.$$

This shows that the HHI defined in equation 1 can be arrived at by equating $\rho = 0$ in equation 2. In the following steps it is shown that this relation holds true even for a perfectly correlated set of assets, i.e. $\rho = 1$. Let us add one more asset, C_3 to our two asset portfolio which makes it equivalent to the portfolio P in the example given in the first section. Then using the MPT, the equivalent of equation 2 for the three asset portfolio P can be written as

$$\frac{\sigma_p^2}{\sigma^2} = c_1^2 + c_2^2 + c_3^2 + 2c_1c_2\rho_{12} + 2c_2c_3\rho_{23} + 2c_3c_1\rho_{31} \tag{3}$$

where ρ_{ij} is the pairwise correlation between the assets i and j. Let us assume that the assets C_1 and C_2 belong to sector S_1 and are perfectly correlated, i.e. $\rho_{12} = 1$, and C_3 belongs to another sector S_2 uncorrelated to S_1, i.e. $\rho_{23} = \rho_{13} = 0$. Also denote the portfolio proportion across the sectors as s_1 and s_2. Clearly, we have $s_1 = c_1 + c_2$ and $s_2 = c_3$. Then, we can write equation 3 as

$$\begin{aligned} \frac{\sigma_p^2}{\sigma^2} &= c_1^2 + c_2^2 + c_3^2 + 2c_1c_2 \\ &= (c_1 + c_2)^2 + c_3^2 \end{aligned}$$

It can be shown that the above equation is similar to the equation for estimating the HHI for the portfolio at the sector level [4] as follows

$$\begin{aligned} HHI &= s_1^2 + s_2^2 \\ &= (c_1 + c_2)^2 + c_3^2 \end{aligned}$$

using $s_1 = c_1 + c_2$ and $s_2 = c_3$. Based on this reasoning we propose that the classic definition of HHI given in equation 1 is a special case of the following generalized HHI (GHHI) equation derived using the MPT

$$GHHI = \sum_i^n c_i^2 + \sum_i^n \sum_{j \neq i}^n 2c_ic_j\rho_{ij} \tag{4}$$

It is interesting to note that for $\rho_{ij} = 1$ the additional term $2c_ic_j\rho_{ij}$ in equation 4 matches with the Modified HHI Delta, $2S_aS_b$, proposed by Bresnahan and Salop [5] to be added to the classic HHI equation to account for a full merger of two companies with market share S_a and S_b. Bresnahan and Salop extended the Cournot Model and derived the Modified HHI (MHHI) to take into account the effect of partial ownerships among firms on the market concentration to measure the impact of changing corporate control on market competition [6]. Salop and O'Brien also derived a general formula for estimating the degree of market concentration for a multi-owner multi-player scenario with partial ownerships

$$MHHI = HHI + \sum_j \sum_{k \neq j} \left(\frac{\sum_i \gamma_{ij}\beta_{ik}}{\sum_i \gamma_{ij}\beta_{ij}} \right)$$

where β_{ij} is the fraction of the firm j owned by the owner i and γ_{ij} is the degree of control that owner i has over the firm j. For the simplest case where the firm j is 100% owned by the firm i, the additional term in the above mentioned equation for MHHI reduces to that in the GHHI for a perfectly correlated pair of assets.

GHHI as a Measure of Risk

Comparing equation 3 and equation 4 we get

$$\begin{aligned} GHHI &= \sum_i^n c_i^2 + \sum_i^n \sum_{j \neq i}^n 2c_ic_j\rho_{ij} \\ &= \frac{\sigma_p^2}{\sigma^2} \end{aligned}$$

where σ_p^2 denotes portfolio risk and σ^2 is the average risk in underlying assets. So, GHHI can also be interpreted as the fraction of the average risk in the underlying assets not mitigated by diversification. It is intuitive that a higher diversification will lead to smaller GHHI which denotes that only a smaller portion of the average risk remains unmitigated.

[4]Note that in equation 1 HHI is defined for the uncorrelated sectors only

A note on correlation

The diversity score should be seen in context of the risk which is being mitigated through portfolio diversification. For example, a credit portfolio diversified across different counterparties in a single sector mitigates the counterparty credit risk but may not mitigate the sector risk. The correlation among assets in equation 3 should correspond to the risk which is being mitigated by portfolio diversification. If the counterparty credit risk is being mitigated through diversification, then credit default correlations among assets should be used to calculate the diversity score. Similarly, if the event risk is being mitigated through diversification then event correlations among diferent assets should be used. Such correlations may be estimated using the historical default and event occurrence data.

Generalized-HHI for Multiple Levels of Sectors and Subsectors

For the following discussion, sectors are denoted by S_j, where $j = 1, 2, ...m$ and counterparties are denoted by C_{ij}, where $i = 1, 2, ..., n_j$, i.e. i^{th} counterparty which belongs to j^{th} sector. Corresponding portfolio proportions across sectors and subsectors can be denoted as s_j and c_{ij}.

The Generalized-HHI (GHHI) formula for a portfolio having exposure to multiple sectors and correlated subsectors may become difficult to handle due to pairwise correlation term for all pairs of the subsectors. However an interesting property allows the GHHI equation 4 to be simplified for a multiple sector portfolio. If the square of proportion of a sector in a portfolio, i.e. s_j^2, is divided by the effective number of subsectors within that sector, the result is the contribution of that sector to the total portfolio GHHI. We discussed earlier that $\frac{1}{HHI}$ can be interpreted as the number of effective units in a sector to which the portfolio is diversified. Using the same interpretation we can simplify the GHHI formula.

Consider the portfolio P with exposure to three counterparties (or subsectors), c_{11}, c_{21}, and c_{12}, which in turn belong to two industries (or sectors), s_1 and s_2. Counterparties in a sector are correlated but the sectors are uncorrelated. The correlation between c_{11} and c_{21} is assumed to be $\rho = 0.5$. Also assume, $c_{11} = 0.4$, $c_{21} = 0.1$, and $c_{12} = 0.5$. Clearly, we then have $s_1 = c_{11} + c_{21} = 0.5$ and $s_2 = c_{12} = 0.5$ The GHHI for the portfolio P using the equation 4 will be

$$
\begin{aligned}
GHHI_P &= [c_{11}^2 + c_{21}^2 + 2c_{11}c_{21}\rho] + c_{12}^2 \\
&= [0.4^2 + 0.1^2 + 2 \times 0.4 \times 0.1 \times 0.5] + 0.5^2 \\
&= [0.21] + 0.25 = 0.47.
\end{aligned}
$$

The contribution of sector S_1, calculated in the square bracket above, is 0.21. This can also be calculated by dividing the square of proportion of S_1, i.e. s_1^2, by the effective number of subsectors in S_1. We know that the effective number can also be obtained by the inverse of HHI (or GHHI if entities are correlated). So, the contribution of S_1 can be calculated by multiplying s_1^2 by the GHHI of S_1. In sector S_1, the proportions of c_{11} and c_{21} are 80% and 20%, and the correlation is 0.5. So, the sector S_1 GHHI will be

$$
\begin{aligned}
GHHI_{S_1} &= 0.8^2 + 0.2^2 + 2 \times 0.8 \times 0.2 \times 0.5 \\
&= 0.84
\end{aligned}
$$

So, the contribution of S_1 to the portfolio GHHI is given by $s_1^2 \times GHHI_{S_1} = 0.5^2 \times 0.84 = 0.21$, which is same as obtained earlier. Similarly, the contribution of S_2 to the portfolio GHHI is given by $s_2^2 \times GHHI_{S_2} = 0.5^2 \times 1.0 = 0.25$ [5]. So, we can write $GHHI_P$ equation as

[5] Note that $GHHI_{S_2} = 1$ as there is only one subsector under S_2

$$GHHI_P = s_1^2 \times GHHI_{S_1} + s_2^2 \times GHHI_{S_2}$$

and more generally, we can rewrite equation 4 as a recursive equation:

$$GHHI = \sum_{j}^{n} s_j^2 \times GHHI_{S_j} \tag{5}$$

In other words if the portfolio P is in turn a part of a larger portfolio $\mathbf{P}$, which also includes other smaller portfolios Q and R, then the GHHI for the larger portfolio $\mathbf{P}$ will be

$$GHHI_{\mathbf{P}} = p^2 \times GHHI_P + q^2 \times GHHI_Q + r^2 \times GHHI_R$$

where p, q and r are the proportions of P, Q and R in $\mathbf{P}$. Àvila et. al. also arrived at a similar recursive equation for the classic HHI measure [2].

Estimating Diversity Scores for Hypothetical Credit Portfolios using GHHI

As an illustration the HHI and GHHI are estimated for four hypothetical credit portfolios, A, B, C, and D, with exposure in 12 counterparties across three sectors S_1, S_2 and S_3. The counterparties are pairwise correlated within a sector with correlations ρ_1, ρ_2 and ρ_3. It is assumed that sectors are pairwise uncorrelated. Column 4 to 6 of Table 1 show the exposure of the four portfolios in different counterparties. Portfolio A is a seemingly perfectly diversified portfolio with equal exposure to all the available counterparties. In fact, a comparison based on the classic HHI yields a similar conclusion. However, it is shown using GHHI that a more diversified portfolio can be created taking into account the correlation among the counterparties in a sector.

The last two rows of Table 1 show the calculated value of diversity scores of each portfolio as well as the effective number of counterparties in each portfolio. Based on HHI, the degree of diversification of the portfolios follows the order: $A > B = C = D$. Whereas GHHI takes into account the correlation and provides a more accurate order: $D > A > C > B$, i.e. a portfolio with exposures skewed towards a highly correlated sector, such as S_3, will have lower risk mitigation.

Conclusion

The paper proposes a more effective and general measure of portfolio diversity score in the form of GHHI, estimated using equation 4 which takes into account multiple levels of sectors as well as correlation among them. GHHI can be used by banks, financial institutions and other credit portfolio managers to estimate and compare the degree of diversification in credit portfolios as well as the fraction of average risk mitigated by such diversification. The paper contributes to the existing body of research work for developing efficient measures to quantify and monitor concentration risk in credit portfolios of large banks and financial institutions.

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Sector S_j	**Correlation** ρ_j	**Counterparty** C_{ij}	**A**	**B**	**C**	**D**
S_1	$\rho_1 = 0.05$	C_{11}	8.33%	2.5%	2.5%	15%
		C_{21}	8.33%	2.5%	2.5%	15%
		C_{31}	8.33%	2.5%	2.5%	15%
		C_{41}	8.33%	2.5%	2.5%	15%
S_2	$\rho_1 = 0.25$	C_{12}	8.33%	7.5%	15%	7.5%
		C_{22}	8.33%	7.5%	15%	7.5%
		C_{32}	8.33%	7.5%	15%	7.5%
		C_{42}	8.33%	7.5%	15%	7.5%
S_3	$\rho_1 = 0.50$	C_{13}	8.33%	15%	7.5%	2.5%
		C_{23}	8.33%	15%	7.5%	2.5%
		C_{33}	8.33%	15%	7.5%	2.5%
		C_{43}	8.33%	15%	7.5%	2.5%
		$HHI =$	0.0833	0.115	0.115	0.115
		$1/HHI =$	12	8.70	8.70	8.70
		$GHHI =$	0.150	0.267	0.217	0.149
		$1/GHHI =$	6.67	3.74	4.62	6.71

Table 1: Comparison of HHI and GHHI for hypothetical portfolios